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Double Soliton Solutions of Belinsky–Zakharov Equation Related to the Self-Dual *SU*(*N*) Gauge Fields

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The Belinsky–Zakharov inverse scattering method is extended to a double high-dimension form. It is associated with the self-dual $SU(N)$ gauge fields. With the harmonic function method, some scattering wave functions can be directly obtained. The one-soliton solution and general n-soliton solution are given.

1. INTRODUCTION

Techniques for generating new solutions of the gauge field are quite important. Among the techniques for the generation of solutions, the inverse scattering method (ISM) developed by Belinsky and Zakharov (BZ) (1978, 1979) is one of the most effective. Letelier (1982) used the ISM to find the soliton solutions of the self-dual $SU(2)$ gauge field and subsequently studied the $SU(N)$ case (Letelier, 1986). Papadopoulos (1985) studied the $SU(3)$ gauge field. However, in these methods only the ordinary complex numbers (with the imaginary unit i, $i^2 = -1$) are used, and many of new solutions are lost. Zhong (1985, 1988) suggested a double-complex function, double-inverse scattering method, and used this method to generate new solutions of the self-dual $SU(2)$ gauge field (Zhong, 1988).

The organization of this paper is as follows: in Section 2 we present the double ISM for the high-dimension BZ equation. In Section 3 we give four particular cases of one-soliton solutions. Section 4 gives the n-soliton solution.

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2. DOUBLE INVERSE SCATTERING METHOD

The general double complex function method has been discussed by Zhong (1985, 1988, 1989). Here we use only the relevant results. Let J denote the double imaginary unit, i.e., $J = i$ ($i^2 = -1$) or $J = \varepsilon$ ($\varepsilon^2 = +1$, $\varepsilon \neq \pm 1$). When all a_n are real numbers and $\Sigma_{n=0}^{\infty} |a_n|$ is a convergent series, then $a(J) = \Sigma_{n=0}^{\infty} a_n J^{2n}$ is a double real number. If $a(J)$ and $b(J)$ both are double real numbers, then $Z(J) = a(J) + J \cdot b(J)$ is a double complex number denoted by $Z_C = Z(J = i) = a_C + ib_C$, $Z_H = Z(J = \varepsilon) = a_H + \varepsilon b_H$. In addition, we define the commutation operator "$\circ$" of the imaginary units as : $J \to \mathring{J}$, $\mathring{i} = \varepsilon$, $\mathring{\varepsilon} = i$.

The self-duality $SU(N)$ gauge field equation is

$$\partial_{\bar{\xi}}(g_\xi g^{-1}) + \partial_{\bar{\zeta}}(g_\zeta g^{-1}) = 0 \tag{1}$$

where $\xi = (1/\sqrt{2})(x + iy)$, $\zeta = (1/\sqrt{2})(z - ix_4)$.

Let the matrix g be a function only of $r = (2\xi\bar{\xi})^{1/2}$ and $z = (\zeta + \bar{\zeta})/2^{1/2}$; then equation (1) reads

$$\partial_r(rg_r g^{-1}) + \partial_z(rg_z g^{-1}) = 0, \qquad \det g = \pm 1, \quad g = g^+ \tag{2}$$

where the subscripts r and z denote partial differentiation, and g is an $N \times N$ Hermitian matrix with unit determinant. One can have an even number of solitons by a physical seed g when $\det g = +1$, and when $\det g = -1$, one can get an odd number of solitons by a nonphysical seed g.

We consider the following double BZ equation:

$$\partial_r[r\partial_r g(J) \cdot g^{-1}(J)] + \partial_r[r\partial_z g(J) \cdot g^{-1}(J)] = 0 \tag{3}$$

$$\det g = -J^2, \qquad g = g^+$$

where $g(J) = g_{ab}(J)$, $g_{ab}(J) = A_{ab}(r, z) + JB_{ab}(r, z)$. When $J = i$, (3) is just the $SU(N, C)$ gauge field; when $N = 2$, it is the self-dual $SU(2)$ gauge field studied by Zhong (1988); when $J = \varepsilon$, according to Zhong (1985), since $SU(N, H)$ is isomorphic to $SL(N, R)$, this equation must be associated with the $SL(N, R)$ gauge field. But it is written out by the hyperbolic complex matrix $g(J = \varepsilon)$.

According to Zhong (1988) and Letelier (1986), we obtain the following double Lax pair:

$$\left(\partial_r + \frac{2\lambda r}{r^2 + \lambda^2}\partial_\lambda\right)\Psi_0(J) = \frac{rU_0(J) + \lambda W_0(J)}{r^2 + \lambda^2}\Psi_0(J)$$
$$\left(\partial_z - \frac{2\lambda^2}{r^2 + \lambda^2}\partial_\lambda\right)\Psi_0(J) = \frac{rW_0(J) - \lambda U_0(J)}{r^2 + \lambda^2}\Psi_0(J) \tag{4}$$

$$\Psi_0(\lambda = 0; J) = g_0(J) \tag{5}$$

where $U_0(J) = r\partial_r g_0(J)\cdot g_0^{-1}(J)$, $W_0(J) = r\partial_z g_0(J)\cdot g_0^{-1}(J)$, and $\Psi_0(J) = \Psi_0(\lambda, r, z; J) = A(\lambda, r, z; J) + iB(\lambda, r, z; J)$ is an $N \times N$ double ordinary complex function matrix. λ is a double ordinary complex spectral parameter, $A(J)$ and $B(J)$ are double real matrices, and $g_0(J)$ (seed solution) is a known solution of equation (3). If $\Psi_0(J)$ has been obtained, the double n-soliton solution $g_n(J)$ of (3) can be obtained as follows:

$$\begin{aligned}
g_n(J) &= \left|\det g_n'(J)\right|^{-1/N} g_n'(J) \\
[g_n'(J)]_{ab} &= [g_0(J)]_{ab} - \sum_{k,l=1}^{n} \frac{\overline{N}_a^{(l)}(J)[\Gamma^{-1}(J)]_{lk} N_b^{(k)}(J)}{\mu_k(J)\overline{\mu}_l(J)} \\
[\Gamma(J)]_{kl} &\equiv \frac{m_a^{(k)}(J)[g_0(J)]_{ab}\overline{m}_b^{(l)}(J)}{\mu_k(J)\overline{\mu}_l(J) + r^2} = [\overline{\Gamma}(J)]_{lk} \\
N_a^{(k)}(J) &\equiv m_b^{(k)}(J)[g_0(J)]_{ba} \\
m_a^{(k)}(J) &\equiv m_{0b}^{(k)}(J)[Q^{(k)}(J)]_{ba} \\
Q^{(k)}(J) &\equiv \Psi_0^{-1}(J)\big|_{\lambda=\mu_k(J)} \\
\det g_n'(J) &= (-1)^n r^{2n}\left(\prod_{i=1}^{n} \left|\mu_l\right|^{-2}\right)\det g_0(J)
\end{aligned} \tag{6}$$

where the $m_{0b}^{(k)}(J)$ are arbitrary double complex constants, and a and b run from 1 to N. In addition,

$$\partial_r\mu_k(J) = \frac{2r\mu_k(J)}{r^2 + \mu_k^2(J)}, \qquad \partial_z\mu_k(J) = \frac{2\mu_k^2(J)}{r^2 + \mu_k^2(J)}$$

$$\mu_k(J) = \mu_k(r, z; J) = \alpha_k(J) - z \pm [(\alpha_k - z)^2 + r^2]^{1/2} \qquad (k = 1, 2, \ldots, n) \tag{7}$$

where the $\alpha_k(J)$ are double real constants.

Generally speaking, the key step of ISM is to find a suitable wave function Ψ_0. However, this usually is very difficult. We find that for some kinds of seed solutions, $\Psi_0(J)$ can be directly obtained by a method similar to that of Gao and Zhong (1992).

From (7), we have

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)\ln \mu_k(J) = 0 \tag{8}$$

$$\left.\frac{\partial_r\mu_k(J)}{2\mu_k(J)}\right|_{\mu_k(J)\to 0} = \frac{1}{r}, \qquad \left.\frac{\partial_z\mu_k(J)}{2\mu_k(J)}\right|_{\mu_k(J)\to 0} = 0 \tag{9}$$

In the formulas (6) the matrix $\Psi_0(J)$ appears only in the form $\Psi_0(J)\big|_{\lambda=\mu_k(J)}$. Thus, to construct the soliton solutions we only need

$$\Psi_{0k}(J)\big|_{\mu_k(J)\to 0} = g_0(J) \tag{10}$$

Therefore, we let

$$Y_k[\phi, \mu_k(J)] \equiv \frac{1}{2}\int \frac{r}{\mu_k(J)}\,[(\partial_r\mu_k(J)\cdot\partial_r\phi - \partial_z\mu_k(J)\cdot\partial_z\phi)\; dr + (\partial_z\mu_k(J)\cdot\partial_r\phi + \partial_r\mu_k(J)\cdot\partial_z\phi)\; dz] \tag{11}$$

Then

$$Y_k[\phi, \mu_k(J)]\big|_{\mu_k(J)\to 0} = \phi \tag{12}$$

if $\phi = \phi(r, z)$ is a harmonic function, i.e., ϕ satisfies

$$\nabla^2\phi(r, z) = 0 \tag{13}$$

From this we have the following theorem:

Theorem. If the seed solution $g_0(r, z; J)$ is dependent on r and z through harmonic functions $\phi_1(r)$, $\phi_2(r, z)$, . . . , $\phi_s(r, z)$ ($s \geq 1$), i.e., $g_0(r, z; J) = g_0(\phi_1, \phi_2, \ldots, \phi_s; J)$, and the following condition is satisfied:

$$\frac{\partial}{\partial\phi_j}\left[\left(\frac{\partial}{\partial\phi_i}\, g_0(\phi_1, \phi_2, \ldots\ \phi_s; J)\right)\cdot g_0^{-1}(\phi_1, \phi_2, \ldots, \phi_s; J)\right] = 0 \qquad (i, j = 1, 2, \ldots, s) \tag{14}$$

then the corresponding scattering wave function can be directly obtained as

$$\Psi_{0k}(J) = g_0(\phi_1 \to Y_1[\phi_1, \mu_k(J)],\ \phi_2 \to Y_2[\phi_2, \mu_k J)],\ \ldots,\ \phi_s \to Y_s[\phi_s, \mu_k(J)]\,;\ J) \tag{15}$$

The proof of this theorem is similar to Gao and Zhong (1992).

3. THE ONE-SOLITON SOLUTIONS

As an example we shall compute the one-soliton solutions associated to particular $SU(7)$ seed solutions. We take the seed solution of (3) given by

$$(g_0)_{ab} = \begin{cases} -J^2 \exp\phi_a & \text{when}\quad a=b=1,2,\ldots, s \\ \phi_a & \text{when}\quad a=b=s+1, s+3, \ldots, N-1 \\ J^2 \exp(Jc_a) & \text{when}\quad a=b-1=s+1, s+3, \ldots, N-1 \\ -J^2\exp(-Jc_{a-1}) & \text{when}\quad a=b+1=s+2, s+4, \ldots, N \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where the c_a are constants, s is a number such that $0 \leq s \leq N$, and $(N - s)/2$ must be an integer; the ϕ_a are harmonic functions, i.e.,

$$\nabla^2 \phi_a(r, z) = 0 \tag{17}$$

and

$$\sum_{a=1}^{n} \phi_a = 0 \tag{18}$$

and

$$\exp(J\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} (J\theta)^n = c(J\theta) + J \cdot s(J\theta) \tag{19}$$

the double sine $s(J\theta)$ and the double cosine $c(J\theta)$ are respectively defined as

$$s(J\theta) = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} (J\theta)^{2n}\theta, \qquad c(J\theta) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (J\theta)^{2n} \tag{20}$$

where θ is a real number; therefore $s(i\theta) = \sin\theta$, $s(\varepsilon\theta) = \text{sh}\,\theta$, $c(i\theta) = \cos\theta$, $c(c\theta) = \text{ch}\,\theta$.

The determinant associated to (16) is

$$\det g_0(J) = (-1)^{(N-s)/2}(-J^2) \tag{21}$$

For the $SU(7)$ case we have four possible s values: $s = 7, 5, 3$, and 1. The corresponding matrices $g_0(J)$ are

$$g_0(J) = \text{diag}(-J^2e^{\phi_1}, -J^2e^{\phi_2}, -J^2e^{\phi_3}, -J^2e^{\phi_4}, -J^2e^{\phi_5}, -J^2e^{\phi_6}, -J^2e^{\phi_7}) \tag{22}$$

$$g_0(J) = \begin{pmatrix} -J^2e^{\phi_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -J^2e^{\phi_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -J^2e^{\phi_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -J^2e^{\phi_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -J^2e^{\phi_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_6 & \mathring{J}^2e^{Jc_6} \\ 0 & 0 & 0 & 0 & 0 & -J^2e^{(-Jc_6)} & 0 \end{pmatrix} \tag{23}$$

$$g_0(J) = \begin{pmatrix} -J^2e^{\phi_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -J^2e^{\phi_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -J^2e^{\phi_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_4 & \mathring{J}^2e^{Jc_4} & 0 & 0 \\ 0 & 0 & 0 & -J^2e^{(-Jc_4)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_6 & \mathring{J}^2e^{Jc_6} \\ 0 & 0 & 0 & 0 & 0 & -J^2e^{(-Jc_6)} & 0 \end{pmatrix} \tag{24}$$

$$g_0(J) = \begin{pmatrix} -J^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_2 & \mathring{J}^2e^{Jc_2} & 0 & 0 & 0 & 0 \\ 0 & -J^2e^{(-Jc_2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_4 & \mathring{J}^2e^{Jc_4} & 0 & 0 \\ 0 & 0 & 0 & -J^2e^{(-Jc_4)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_6 & \mathring{J}^2e^{Jc6} \\ 0 & 0 & 0 & 0 & 0 & -J^2e^{(-Jc_6)} & 0 \end{pmatrix} \tag{25}$$

By the theorem, the corresponding wave function is

$$(\Psi_0)_{ab} = \begin{cases} -J^2 \exp Y_a^{(k)} & \text{when} \quad a=b=1,2,\dots,s \\ Y_a^{(k)} & \text{when} \quad a=b=s+1,s+3,\dots,N-1 \\ \mathring{J}^2 \exp(Jc_a) & \text{when} \quad a=b-1=s+1,s+3,\dots,N-1 \\ -J^2\exp(-Jc_{a-1}) & \text{when} \quad a=b+1=s+2,s+4,\dots,N \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

Note that (11) and (18) imply that

$$\sum_{a=1}^{s} Y_a^{(k)} = 0 \tag{27}$$

For the diagonal case (22) we get

$$g_{ab(1)} = \left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(1)}^{-1} \times \exp \phi_a\left[\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)\left|q_a\right|^2 \exp(\phi_a - 2 \operatorname{Re} Y_a^{(1)}) - \left|\frac{\mu_1}{r}\right|^2 \Delta_{(1)}\right] \tag{28}$$

when $a = b$, and for $a \neq b$,

$$g_{ab(1)} = \left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(1)}^{-1}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)\overline{q}_a q_b \exp(\phi_a + \phi_b - \overline{Y}_a^{(1)} - \overline{Y}_b^{(1)}) \tag{29}$$

where

$$\Delta_{(1)} = \sum_{b=1}^{7} \left|q_b\right|^2 \exp(\phi_b - 2 \operatorname{Re} Y_b^{(1)}), \qquad q_a^{(1)} = m_{0a}^{(1)} \tag{30}$$

For the case (23) we get, when a, $b \leq 5$, similar formulas to (26), but now

$$\Delta_{(2)} = -2J^2 \operatorname{Re}(q_6\overline{q}_7 \exp Jc_6) + \left|q_7\right|^2(\phi_6 - 2 \operatorname{Re} Y_6^{(1)}) - J^2 \sum_{b=1}^{5} \left|q_b\right|^2 \exp(\phi_b - 2 \operatorname{Re} Y_b^{(1)}) \tag{31}$$

For $a \leq 5$ and $b = 6$, we have

$$g_{a6(2)} = J^2\left|\frac{r}{\mu_1}\right|^{12/7}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)\Delta_{(2)}^{-1}[q_a q_6 \exp(\phi_a - \overline{Y}_a^{(1)}) - J^2\overline{q}_a q_7(\phi_6 - Y_6) \exp(\phi_a - Y_a - Jc_6)] \tag{32}$$

and for $a \leq 5$ and $b = 7$

$$g_{a7(2)} = J^2\left|\frac{r}{\mu_1}\right|^{12/7}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)\Delta_{(2)}^{-1}\overline{q}_a q_7 \exp(\phi_a - \overline{Y}_a^{(1)}) \tag{33}$$

Also,

$$g_{66(2)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(2)}^{-1}\left[\left|\frac{\mu_1}{r}\right|^2 \Delta_{(2)}\phi_6 - \left(1 + \left|\frac{\mu_1}{r}\right|^2\right)\left|q_6 \exp Jc_6 - J^2 q_7(\phi_6 - Y_6^{(1)})\right|^2\right] \tag{34}$$

$$g_{67(2)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(2)}^{-1}\left\{\left|\frac{\mu_1}{r}\right|^2 \Delta_{(2)} \exp Jc_6\right.$$

$$- \left(1 + \left|\frac{\mu_1}{r}\right|^2\right)[\bar{q}_6 q_7 - J^2 |q_7|^2 (\phi_6 - Y_6^{(1)}) \exp Jc_6]\Big\} \tag{35}$$

$$g_{77(2)} = -J^2 \left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(2)}^{-1} \left(1 + \left|\frac{\mu_1}{r}\right|^2\right) |q_7|^2 \tag{36}$$

The other entries of the matrix can be obtained from the property $g = g^{+}$.

For the (24) seed solution we have when $a, b \leq 3$ similar to (26), but in this case

$$\Delta_{(3)} = -J^2 \sum_{b=1}^{3} |q_b|^2 \exp(\phi_b - 2\,\mathrm{Re}\,Y_b^{(1)}) + |q_5|^2 (\phi_4 - 2\,\mathrm{Re}\,Y_4^{(1)})$$
$$+ |q_7|^2 (\phi_6 - 2\,\mathrm{Re}\,Y_6^{(1)}) - 2J^2\,\mathrm{Re}(q_4 \bar{q}_5 \exp Jc_4 + q_6 \bar{q}_7 \exp Jc_6) \tag{37}$$

and when $a \leq 3$, $b = 4$

$$g_{a4(3)} = -J^2 \left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(3)}^{-1} \left(1 + \left|\frac{\mu_1}{4r}\right|^2\right)$$
$$\times \exp(\phi_a - \bar{Y}_a^{(1)})\,[q_a q_4 - J^2 q_a q_5 (\phi_4 - Y_4^{(1)}) \exp(-Jc_4)] \tag{38}$$

$$g_{a5(3)} = -J^2 \left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(3)}^{-1} \left(1 + \left|\frac{\mu_1}{r}\right|^2\right) q_a q_5 \exp(\phi_a - \bar{Y}_a^{(1)}) \tag{39}$$

and

$$g_{44(3)} = -J^2 \left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(3)}^{-1} \Bigg[\left|\frac{\mu_1}{r}\right|^2 \Delta_{(3)} \phi_4 - \left(1 + \left|\frac{\mu_1}{r}\right|^2\right) \Big| q_4 \exp Jc_4$$
$$- J^2 q_5 (\phi_1 - Y_4^{(1)}) \Big|^2 \Bigg] \tag{40}$$

$$g_{45(3)} = -J^2 \left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(3)}^{-1} \Bigg\{ -J^2 \left|\frac{\mu_1}{r}\right|^2 \Delta_{(3)} \exp Jc_2 - \left(1 + \left|\frac{\mu_1}{r}\right|^2\right) [\bar{q}_4 q_5$$
$$- J^2 |q_5|^2 (\phi_4 - Y_4^{(1)}) \exp Jc_4] \Bigg\} \tag{41}$$

$$g_{55(3)} = -J^2 \left|\frac{r}{\mu_1}\right|^{12/7} \Delta_{(3)}^{-1} \left(1 + \left|\frac{\mu_1}{4r}\right|^2\right) |q_5|^2 \tag{42}$$

For $a \leq 3$ and $b = 6$

$$g_{a6(3)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(3)}^{-1}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right) \times \exp(\phi_a - \overline{Y}_a^{(1)})[\overline{q}_a q_6 - J^2\overline{q}_a q_7(\phi_6 - Y_6^{(1)})\exp(-Jc_6)] \tag{43}$$

and

$$g_{46(3)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(3)}^{-1}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)[\overline{q}_1 - J^2 q_5(\phi_4 - \overline{Y}_1^{(1)})\exp Jc_4] \times [q_6 - J^2 q_7(\phi_6 - Y_6^{(1)})\exp(-Jc_6)] \tag{44}$$

$$g_{56(3)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(3)}^{-1}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)[q_5 q_6 - J^2 q_5 q_7(\phi_6 - Y_6^{(1)})\exp(-Jc_6)] \tag{45}$$

$$g_{66(3)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(3)}^{-1}\left[\left|\frac{\mu_1}{r}\right|^2\Delta_{(3)}\phi_6 - \left(1 + \left|\frac{\mu_1}{r}\right|^2\right)\left|q_6\exp Jc_6 - J^2 q_7(\phi_6 - Y_6^{(1)})\right|^2\right] \tag{46}$$

For $a \leq 3$ and $b = 7$

$$g_{a7(3)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(3)}^{-1}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)\overline{q}_a q_7\exp(\phi_a - \overline{Y}_a^{(1)}) \tag{47}$$

and

$$g_{47(3)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(3)}^{-1}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)[\overline{q}_1 q_7 - J^2\overline{q}_5 q_7(\phi_4 - \overline{Y}_4^{(1)})\exp Jc_4] \tag{48}$$

$$g_{57(3)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(3)}^{-1}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)\overline{q}_5 q_7 \tag{49}$$

$$g_{67(3)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(3)}^{-1}\left\{-J^2\left|\frac{\mu_1}{r}\right|^2\Delta_{(3)}\exp Jc_6 + \left(1 + \left|\frac{\mu_1}{r}\right|^2\right)[\overline{q}_6 q_7 - J^2|q_7|^2(\phi_6 - \overline{Y}_6^{(1)})\exp Jc_6]\right\} \tag{50}$$

$$g_{77(3)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(3)}^{-1}\left(1 + \left|\frac{\mu_1}{r}\right|^2\right)|q_7|^2 \tag{51}$$

Finally, for the (25) seed solution we have

$$g_{11(4)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(4)}^{-1}\left[\left|\frac{\mu_1}{r}\right|^2\Delta_{(4)} + J^2|q_1|^2\left(1+\left|\frac{\mu_1}{r}\right|^2\right)\right] \tag{52}$$

$$g_{12(4)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(4)}^{-1}\left(1+\left|\frac{\mu_1}{r}\right|^2\right)\left[\bar{q}_1q_2 - J^2\bar{q}_1q_3(\phi_2 - Y_2^{(1)})\exp(-Jc_2)\right] \tag{53}$$

$$g_{13(4)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(4)}^{-1}\left(1+\left|\frac{\mu_1}{r}\right|^2\right)\bar{q}_1q_3 \tag{54}$$

$$g_{14(4)} = g_{12(4)}(2\to 4, 3\to 5) \tag{55}$$

$$g_{15(4)} = g_{13(4)}(3\to 5) \tag{56}$$

$$g_{16(4)} = g_{14(4)}(4\to 6, 5\to 7) \tag{57}$$

$$g_{17(4)} = g_{13(4)}(3\to 5\to 7) \tag{58}$$

$$g_{22(4)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(4)}^{-1}\left[\left|\frac{\mu_1}{r}\right|^2\Delta_{(4)}\phi_2 - \left(1+\left|\frac{\mu_1}{r}\right|^2\right)\left|q_2\exp Jc_2 - J^2q_3(\phi_2 - Y_2^{(1)})\right|^2\right] \tag{59}$$

$$g_{23(4)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(4)}^{-1}\left\{-J^2\left|\frac{\mu_1}{r}\right|^2\Delta_{(4)}\exp Jc_2 - \left(1+\left|\frac{\mu_1}{r}\right|^2\right)\left[\bar{q}_2q_3 - J^2|q_3|^2(\phi_2 - \bar{Y}_2^{(1)})\exp Jc_2\right]\right\} \tag{60}$$

$$g_{24(4)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(4)}^{-1}\left(1+\left|\frac{\mu_1}{r}\right|^2\right)\left[\bar{q}_2 - J^2\bar{q}_3(\phi_2 - \bar{Y}_2^{(1)})\exp Jc_2\right] \times\left[q_4 - J^2q_5(\phi_4 - Y_4^{(1)})\exp(-Jc_4)\right] \tag{61}$$

$$g_{25(4)} = -J^2\left|\frac{r}{\mu_1}\right|^{12/7}\Delta_{(4)}^{-1}\left(1+\left|\frac{\mu_1}{r}\right|^2\right)\left[\bar{q}_2q_5 - J^2\bar{q}_3q_5(\phi_2 - \bar{Y}_2^{(1)})\exp Jc_2\right] \tag{62}$$

$$g_{26(4)} = g_{24(4)}(4\to 6,\ 5\to 7) \tag{63}$$

$$g_{27(4)} = g_{25(4)}(5 \to 7) \tag{64}$$

$$g_{33(4)} = g_{13(4)}(1 \to 3) \tag{65}$$

$$g_{34(4)} = g_{12(4)}(1 \to 3,\ 2 \to 4,\ 3 \to 5) \tag{66}$$

$$g_{35(4)} = g_{13(4)}(1 \to 3,\ 3 \to 5) \tag{67}$$

$$g_{36(4)} = g_{12(4)}(1 \to 3,\ 2 \to 4 \to 6,\ 3 \to 5 \to 7) \tag{68}$$

$$g_{37(4)} = g_{17(4)}(1 \to 3) \tag{69}$$

$$g_{44(4)} = g_{22(4)}(2 \to 4,\ 3 \to 5) \tag{70}$$

$$g_{45(4)} = g_{23(4)}(2 \to 4,\ 3 \to 5) \tag{71}$$

$$g_{46(4)} = g_{24(4)}(2 \to 4,\ 3 \to 5,\ 5 \to 7) \tag{72}$$

$$g_{47(4)} = g_{25(4)}(2 \to 4,\ 3 \to 5,\ 5 \to 7) \tag{73}$$

$$g_{55(4)} = g_{33(4)}(3 \to 5) \tag{74}$$

$$g_{56(4)} = g_{12(4)}(1 \to 3 \to 5,\ 2 \to 4 \to 6,\ 3 \to 5 \to 7) \tag{75}$$

$$g_{57(4)} = g_{13(4)}(1 \to 3 \to 5,\ 3 \to 5 \to 7) \tag{76}$$

$$g_{66(4)} = g_{22(4)}(2 \to 4 \to 6,\ 3 \to 5 \to 7) \tag{77}$$

$$g_{67(4)} = g_{23(4)}(2 \to 1 \to 6,\ 3 \to 5 \to 7) \tag{78}$$

$$g_{77(4)} = g_{13(4)}(1 \to 3 \to 5 \to 7,\ 3 \to 5 \to 7) \tag{79}$$

where $g_{12(4)}(2 \to 4,\ 3 \to 5)$, etc., means that we let $2 \to 4$ and $3 \to 5$ in the expression (32d), etc. For this case, $\Delta_{(4)}$ is given by

$$\begin{aligned}\Delta_{(4)} = {} & -J^2|q_1|^2 + |q_3|^2(\phi_2 - 2\ \mathrm{Re}\ Y_2^{(1)}) + |q_5|^2(\phi_4 - 2\ \mathrm{Re}\ Y_4^{(1)}) \\ & + |q_7|^2(\phi_6 - 2\ \mathrm{Re}\ Y_6^{(1)}) - 2J^2\ \mathrm{Re}(q_2\bar{q}_3\ 3\mathrm{xp}\ Jc_2 + q_4\bar{q}_5 \exp Jc_4 \\ & + q_6\bar{q}_7 \exp Jc_6)\end{aligned} \tag{80}$$

4. THE *N*-SOLITON SOLUTION

In the general case, to compute the elements of the seed solution $g_n(J)$ we first need to compute $Q^{(k)}$ and $m_a^{(k)}$ as given by (6). We find

$$(Q^{(k)})_{ab} = \begin{cases} -J^2 \exp(-Y_a^{(k)}) & \text{when} \quad a = b = 1, 2, \ldots, s \\ -Y_{a-1}^{(k)} & \text{when} \quad a = b = s+2, s+4, \ldots, N \\ J^2 \exp(Jc_a) & \text{when} \quad a = b-1 = s+1, s+3, \ldots, N-1 \\ -J^2 \exp(-Jc_a) & \text{when} \quad a = b+1 = s+2, s+4, \ldots, N \\ 0 & \text{otherwise} \end{cases} \tag{81}$$

and

$$m_b^{(k)} = \begin{cases} -J^2 m_{0b}^{(k)} \exp(-Y_b^{(k)}) & \text{when} \quad b \leq s \\ -J^2 m_{0b+1}^{(k)} \exp(-Jc_b) & \text{when} \quad b = s+1, s+3, \ldots, N-1 \\ -m_{0b}^{(k)} Y_{b-1}^{(k)} + m_{0b-1}^{(k)} \exp(Jc_{b-1}) & \text{when} \quad b = s+2, s+4, \ldots, N \end{cases} \tag{82}$$

Then we get

$$N_b^{(k)} = \begin{cases} m_{0b}^{(k)} \exp(\phi_b - Y_b^{(k)}) & \text{when} \quad b \leq s \\ m_{0b}^{(k)} - J^2 m_{0b+1}^{(k)} (\phi_b - Y_b^{(k)}) \exp(-Jc_b) & \text{when} \quad b = s+1, s+3, \ldots, N-1 \\ m_{0b}^{(k)} & \text{when} \quad b = s+2, s+4, \ldots, N \end{cases} \tag{83}$$

and

$$\begin{aligned} \Gamma_{kl} = (\mu_k(J)\overline{\mu}_l(J) + r^2)^{-1} \Bigg\{ & \sum_{b=1}^{s} (-J^2) m_{0b}^{(k)} \overline{m}_{0b}^{(l)} \exp(\phi_b - Y_b^{(k)} - \overline{Y}_b^{(l)}) \\ & + \sum_{b=s+1}^{N-1}{}' \; m_{0b+1}^{(k)} \overline{m}_{0b+1}^{(l)} \exp(\phi_b - Y_b^{(k)} - \overline{Y}_b^{(l)}) \\ & + \sum_{b=s+1}^{N-1}{}' \; [(-J^2) m_{0b}^{(k)} \overline{m}_{0b+1}^{(l)} \exp Jc_b - J^2 m_{0b+1}^{(k)} \overline{m}_{0b}^{(l)} \exp(-Jc_b)] \Bigg\} \end{aligned} \tag{84}$$

where Σ' indicates a sum on $b = s + 1, s + 3, \ldots, N - 2$.

One can use these quantities to obtain the n-soliton solution.

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